ASYMPTOTIC STABILITY OF MULTI-SOLITON SOLUTIONS FOR NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the Cauchy problem for the nonlinear Schrödinger equation $i\psi_t = -\triangle\psi + F(|\psi|^2)\psi$, in space dimensions $d \geq 3$, with initial data close to a sum of N decoupled solitons. Under some suitable assumptions on the spectral structure of the one soliton linearizations we prove that for large time the asymptotics of the solution is given by a sum of solitons with slightly modified parameters and a small dispersive term.

0. Introduction

In this paper we consider the nonlinear Schrödinger equation

$$i\psi_t = -\triangle\psi + F(|\psi|^2)\psi, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d,\ d \geq 3. \tag{0.1}$$

For suitable F it possesses important solutions of special form - solitary waves (or, shortly, solitons):

$$e^{i\Phi}\varphi(x - b(t), E),$$

$$\Phi = \omega t + \gamma + \frac{1}{2}x\cdot v,\ b(t) = vt + c,\ E = \omega + \frac{|v|^2}{4} > 0,$$

where ω, $\gamma \in \mathbb{R}$, v, $c \in \mathbb{R}^d$ are constants and φ is a ground state that is a smooth positive spherically symmetric, exponentially decreasing solution of the equation

$$-\triangle\varphi + E\varphi + F(\varphi^2)\varphi = 0. \tag{0.2}$$

Solitary wave solutions are of special importance not only because they are simple and sometimes explicit solutions of evolution equations, but also because of the distinguished role they appear to play in the solution of the initial value problem. This is best known for completely integrable equations like the cubic Schrödinger equation

$$i\psi_t = -\psi_{xx} - |\psi|^2\psi. \tag{0.3}$$

In the general position case the solution of the Cauchy problem for this equation with rapidly decreasing smooth initial data has in $L_2(\mathbb{R})$ the asymptotic behavior

$$\psi \sim \sum_{j=1}^{N} e^{i\Phi_j}\varphi(x - b_j(t), E_j) + e^{il_0 t}f_+, \quad \Phi_j = \omega_j t + \gamma_j + \frac{xv_j}{2},\ b_j = v_j t + c_j$$

where $l_0 = -\partial_x^2$, and f_+ is some function in $L_2(\mathbb{R})$. The number N, the function f_+ and the soliton parameters $(\gamma_j, E_j, v_j, c_j)$ depend on the initial data. Due to possibility of explicitly integrating equation (0.3) with the help of inverse scattering methods, they can be described by effective formulas in terms of initial condition. See, for example, [22] for these results.

Numerical experiments have shown that even without the presence of an inverse-scattering theory, solutions, in general, eventually resolve themselves into an approximate superposition of weakly interacting solitary waves and decaying dispersive waves (see [11], for example). While exact theory confirming the special role of solitary waves as a nonlinear basis with respect to which it is natural to view the solutions in the limit of large time is not generally available, partial indication is provided by stability theory of such waves. A considerable literature has been devoted to the problem of orbital stability of solitons following the work of Benjamin [1], see also [7, 13, 14, 29, 34, 35]. The problem arises in connection with the Cauchy problem for equation (0.1) with initial data of the form

$$\psi\big|_{t=0} = \varphi(x, E_0) + \chi_0, \tag{0.4}$$

where χ_0 is small in the Sobolev space $H^1(\mathbb{R}^d)$. It was shown that under certain additional conditions the solution $\psi(x,t)$, $t \geq 0$ remains close (again in the space $H^1(\mathbb{R}^d)$) to the surface

$$\{e^{i\gamma}\varphi(x-c, E_0),\ \gamma \in \mathbb{R}, c \in \mathbb{R}^d\}.$$

This notion of stability establishes that the shape of the wave is stable, but does not fully resolve the question of what the asymptotic behavior of the system is.

The first asymptotic stability results were obtained by Soffer and Weinstein in the context of the equation

$$i\psi_t = -\triangle\psi + [V(x) + \lambda|\psi|^{m-1}]\psi, \tag{0.5}$$

(see [27, 28] and [30, 31, 32, 33, 36] for the further developments related to this model). The solitons for (0.5) arise as a perturbation of the eigenfunction of the operator $-\triangle + V(x)$ and, in contrast to the case of equation (0.1), they have a fixed center, which simplifies the analysis to some extent.

For the one- dimensional equation

$$i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi \tag{0.6}$$

the asymptotic stability of solitons was studied in the works of Buslaev and author [4, 5]. We considered the Cauchy problem (0.6), (0.4) and proved that in the case where the spectrum of the linearization of equation (0.6) at the initial soliton has the simplest possible structure in some natural sense, the solution ψ has an asymptotic behavior of the form

$$\psi = e^{i\Phi_+}\varphi(x-b_+(t), E_+) + e^{-il_0t}f_+ + o(1), \quad \Phi_+ = \omega_+t + \gamma_+ + \frac{xv_+}{2},\ b_+ = v_+t + c_+$$

as $t \to +\infty$, where the parameters (γ_+, E_+, v_+c_+) of the limit soliton are close to the initial ones $(0, E_0, 0, 0)$ and f_+ is small. Some asymptotic results in the

framework of significantly freer conditions on the linearization were obtained in [5], see also [6]. Recently the analysis of [4, 5, 6] was extended to the multidimensional case (0.1) by Cuccagna [8, 9].

As a natural generalization of the above situation one can consider the case of several weakly interacting solitons. Assume that one has a set of solitons $e^{i\beta_{0j}+i\frac{x\cdot v_{0j}}{2}}\varphi(x-b_{0j},E_{0j})$, $j=1,\dots,N$, that are well separated either in the original space or in Fourier space: for $j\neq k$, either $|v^0_{jk}|$ or $\min_{t\geq 0}|b^0_{jk}(t)|$ is sufficiently large, where $v^0_{jk}=v_{0j}-v_{0k}$, $b^0_{jk}(t)=b_{0j}-b_{0k}+v^0_{jk}t$. In the second case we shall assume that the "collision time" $t^0_{jk}=-\frac{b^0_{jk}(0)\cdot v^0_{jk}}{|v^0_{jk}|^2}$ is "bounded" from above, see subsection 1.4, (1.7) for the exact formulation.

Consider the Cauchy problem for equation (0.1) with initial data close to a sum

$$\sum_{j=1}^{N} e^{i\beta_{0j}+i\frac{x\cdot v_{0j}}{2}}\varphi(x-b_{0j},E_{0j}).$$

If all the linearizations constructed independently from the solitons $\varphi(E_{0j})$ satisfy the spectral conditions introduced in the case of one soliton, one can expect that as $t\to+\infty$ the solution ψ looks like a sum of N soliton with slightly modified parameters plus a small dispersive term. In [23] this was proved in the case $d=1$, $N=2$, see also [19] for the asymptotic stability results for the sums of solitons in the context of KdV type equations. The goal of the present paper is to extend the result of [23] to the multidimensional case $d\geq 3$ (omitting also the restriction N=2). The main new ingredient in the analysis is a combination of the estimates for the linear one soliton evolution obtained by Cuccagna in [8] with the ideas of Hagedorn [15].

The structure of this paper is briefly as follows. It consists of two sections. In the first section we introduce some preliminary objects and state the main result. The second contains the complete proofs of the indicated results, some technical details being removed to the appendices.

1. Background and statement of the results

1.1. Assumptions on F. Consider the nonlinear Schrödinger equation

$$i\psi_t=-\triangle\psi+F(|\psi|^2)\psi,\quad (t,x)\in\mathbb{R}\times\mathbb{R}^d,\ d\geq 3. \tag{1.1}$$

We assume the following.

Hypothesis H0. *F is a smooth function, $F(0)=0$, F satisfies the estimates*

$$F(\xi)\geq -C\xi^q,\quad |F^{(\alpha)}(\xi)|\leq C\xi^{p-\alpha},\ \alpha=0,1,2,$$

where $C>0$, $\xi\geq 1$, $q<\frac{2}{d}$, $p<\frac{2}{d-2}$.

Set $g(\xi)=E\xi+F(\xi^2)\xi$.

Hypothesis H1.

(i) *There exists $\xi_0>0$ such that $g(\xi)>0$ for $\xi<\xi_0$, $g(\xi)<0$ for $\xi>\xi_0$ and $g'(\xi_0)<0$.*

(ii) *There exists $\xi_1>0$ such that $\int_0^{\xi_1}ds g(s)=0$.*

Further assumptions are given in terms of the function

$$I(\xi,\lambda)=-\lambda\xi g'(\xi)+(\lambda+2)g(\xi).$$

We consider ξ_0 of (H1) and assume:

Hypothesis H2. *For any $\xi > \xi_0$ there exists a $\lambda(\xi) > 0$, continuously depending on ξ, such that $I(t, \lambda) \geq 0$ for $0 < t < \xi$ and $I(t, \lambda) \leq 0$ for $t > \xi$.*

We suppose hypotheses (H1,2) to be true for E in some open interval $\mathcal{A} \subset \mathbb{R}_+$.

Under these assumptions equation (0.2) for $E \in \mathcal{A}$, has a unique positive spherically symmetric smooth exponentially decreasing solution $\varphi(x, E)$, see [2, 20]. More precisely, as $|x| \to \infty$

$$\varphi(x, E) \sim Ce^{-\sqrt{E}|x|}|x|^{-\frac{(d-1)}{2}}.$$

This asymptotic estimate can be differentiated any number of times with respect to x and E.

We shall call the functions $w(x, \sigma) = \exp(i\beta + iv\cdot x/2)\varphi(x - b, E)$, $\sigma = (\beta, E, b, v) \in \mathbb{R}^{2d+2}$ by soliton states. $w(x, \sigma(t))$ is a solitary wave solution iff $\sigma(t)$ satisfies the system:

$$\beta' = E - \frac{|v|^2}{4}, \quad E' = 0, \quad b' = v, \quad v' = 0. \tag{1.2}$$

1.2. One soliton linearization. Consider the linearization of equation (1.1) on a soliton $w(x, \sigma(t))$:

$$\psi \sim w + \chi,$$

$$i\chi_t = (-\triangle + F(|w|^2))\chi + F'(|w|^2)(|w|^2\chi + w^2\bar{\chi}).$$

Introducing the function $\vec{f}$:

$$\vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \quad \chi(x, t) = \exp(i\Phi)f(y, t),$$

$$\Phi = \beta(t) + \frac{v \cdot x}{2}, \quad y = x - b(t),$$

one gets

$$i\vec{f}_t = L(E)\vec{f}, \quad L(E) = L_0(E) + V(E), \; L_0(E) = (-\triangle + E)\sigma_3,$$

$$V(E) = V_1(E)\sigma_3 + iV_2(E)\sigma_2, \quad V_1 = F(\varphi^2) + F'(\varphi^2)\varphi^2, \; V_2(E) = F'(\varphi^2)\varphi^2.$$

Here σ_2, σ_3 are the standard Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider L as an operator in $L_2(\mathbb{R}^d \to \mathbb{C}^2)$ defined on the domain where L_0 is self adjoint. L satisfies the relations

$$\sigma_3 L \sigma_3 = L^*, \quad \sigma_1 L \sigma_1 = -L,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The continuous spectrum of $L(E)$ fills up two semi-axes $(-\infty, E]$ and $[E, \infty)$. In addition $L(E)$ may have finite and finite dimensional point spectrum on the real and imaginary axis.

Zero is always a point of the discrete spectrum. One can indicate $d+1$ eigenfunctions

$$\vec{\xi}_0 = \varphi\begin{pmatrix}1\\-1\end{pmatrix}, \quad \vec{\xi}_j = \varphi_{y_j}\begin{pmatrix}1\\1\end{pmatrix},\ j=1,\dots d,$$

and $d+1$ generalized eigenfunctions

$$\vec{\xi}_{d+1} = -\varphi_E\begin{pmatrix}1\\1\end{pmatrix}, \quad \vec{\xi}_{d+1+j} = -\frac{1}{2}y_j\varphi\begin{pmatrix}1\\1\end{pmatrix},\ j=1,\dots d,$$

$$L\vec{\xi}_j = 0, \quad L\vec{\xi}_{d+1+j} = \vec{\xi}_j,\ j=0,\dots,d.$$

Let M be the generalized null space of the operator L. Under assumptions (H0,1,2), the vectors $\vec{\xi}_j,\ j=0,\dots,2d+1$, span the subspace M iff

$$\frac{d}{dE}\|\varphi(E)\|_2^2 \neq 0,$$

see [34, 20, 8].

We shall assume that

Hypothesis H3. *The set $\mathcal{A}_0$ of $E\in\mathcal{A}$ such that*

(i) *zero is the only eigenvalue of the operator $L(E)$, and the dimension of the corresponding generalized null space is equal to $2d+2$;*

(ii) *$\pm E$ is not a resonance for $L(E)$;*

is nonempty.

Obviously, the set $\mathcal{A}_0$ is open.

Remark. $\pm E$ is said to be a resonance of $L(E)$ if there is a solution ψ of the equation $(L(E)\mp E)\psi = 0$ such that $<x>^{-s}\psi\in L_2$ for any $s>1/2$ but not for $s=0$. $\pm E$ can never be a resonance if $d\geq 5$, see lemma A4.3.

Consider the evolution operator e^{-itL}. One has the following proposition.

Proposition 1.1. *For $E\in\mathcal{A}_0$ and any $x_0, x_1\in\mathbb{R}^d$,*

$$\| \langle x-x_0\rangle^{-\nu_0} e^{-iL(E)t}\hat{P}(E)f\|_2 \leq C\,\langle t\rangle^{-d/2}\,\|\langle x-x_1\rangle^{\nu_0} f\|_2, \quad \nu_0>\frac{d}{2}, \tag{1.3}$$

where $\hat{P}(E)$ is the spectral projection onto the subspace of the continuous spectrum of $L(E)$:

$$\operatorname{Ker}\hat{P} = M, \quad \operatorname{Ran}\hat{P} = (\sigma_3 M)^\perp.$$

The constant C here is uniform with respect to $x_0, x_1\in\mathbb{R}^d$ and E in compact subsets of $\mathcal{A}_0$.

This proposition is an immediate consequence of the L_p- L_q estimates of $e^{-iLt}\hat{P}$ proved by Cuccagna [8]. For the sake of completeness we sketch the proof of (1.3) in appendix 4.

1.3. The nonlinear equation. We formulate here the necessary facts about the Cauchy problem for equation (1.1) with initial data in $H^1(\mathbb{R}^d)$.

Proposition 1.2. *Suppose that F satisfies* (H0). *Then the Cauchy problem for equation* (1.1) *with initial data $\psi(x,0)=\psi_0(x)$, $\psi_0\in H^1(\mathbb{R}^d)$ has a unique solution ψ in the space $C(\mathbb{R}\to H^1)$, and ψ satisfies the conservation laws*

$$\int dx|\psi|^2 = const, \quad H(\psi)\equiv\int dx[|\nabla\psi|^2+U(|\psi|^2)] = const,$$

where $U(\xi)=\int_0^\xi dsF(s)$. Furthermore, for all $t\in\mathbb{R}$

$$\|\psi(t)\|_{H^1}\le c(\|\psi_0\|_{H^1})\|\psi_0\|_{H^1},$$

where $c:\mathbb{R}_+\to\mathbb{R}_+$ is a smooth function.

The assertion stated here can be found in [10, 11], for example.

1.4. Description of the problem. Consider the Cauchy problem for equation (1.1) with initial data

$$\psi|_{t=0}=\psi_0\in H^1\cap L_1, \quad \psi_0=\sum_{j=1}^N w(\cdot,\sigma_{0j})+\chi_0, \tag{1.4}$$

$$\sigma_{0j}=(\beta_{0j},E_{0j},b_{0j},v_{0j}), \quad \min_{j\neq k}|v^0_{jk}|\ge v_0>0. \tag{1.5}$$

Here $v^0_{jk}=v_{0j}-v_{0k}$. Set $b^0_{jk}=b_{0j}-b_{0k}$, $j\neq k$. Write b^0_{jk} as the sum

$$b^0_{jk}=r^0_{jk}-t^0_{jk}v^0_{jk}, \quad r^0_{jk}\cdot v^0_{jk}=0,\ t^0_{jk}=-\frac{b^0_{jk}\cdot v^0_{jk}}{|v^0_{jk}|^2}. \tag{1.6}$$

For $j\neq k$ we define the effective small parameter ϵ_{jk}:

$$\epsilon_{jk}=\begin{cases}(\min_{t\ge0}|b^0_{jk}(t)|+|v^0_{jk}|)^{-1}, & \text{if } t^0_{jk}\le\kappa<r^0_{jk}>,\\ |v^0_{jk}|^{-1} & \text{otherwise,}\end{cases} \tag{1.7}$$

where $b^0_{jk}(t)=b^0_{jk}+tv^0_{jk}$, κ is a fixed positive constant.

Assume that

(T1) $\epsilon\equiv\max_{j\neq k}\epsilon_{jk}$ *is sufficiently small*[1];

(T2) $E_{0j}\in\mathcal{A}_0$, $j=1,\dots,N$.

Our goal is to describe the asymptotic behavior of the solution ψ as $t\to+\infty$, provided χ_0 is sufficiently small in the following sense:

(T3) *for some m', $\frac{1}{m}+\frac{1}{m'}=1$, $m\ge 2p+2$, $\frac{4}{d}+2<m<\frac{4}{d-2}+2$ if $d\ge4$, $4\le m<\frac{4}{d-2}+2$ if $d=3$, the norm*

$$\mathcal{N}=\|\chi_0\|_1+\|\hat\chi_0\|_{m'}$$

is sufficiently small.

Here $\hat\chi_0$ stands for the Fourier transform of χ_0.

Our main result is given by the following theorem.

[1] "Sufficiently small (large)" assumes constants that depend only on v_0, κ and E_{0j}, $j=1,\dots,N$.

Theorem 1.1. *For $t \geq 0$ the solution ψ of* (1.1), (1.4) *admits the representation*

$$\psi(t) = \sum_{j=1}^{N} w(\cdot, \sigma_j(t)) + \chi(t), \quad \sigma_j(t) = (\beta_j(t), E_j(t), b_j(t), v_j(t)),$$

where $|E_j(t) - E_{0j}|$, $|v_j(t) - v_{0j}|$, $j = 1, \dots, N$, $\|\chi(t)\|_{L_2 \cap L_m}$ are small uniformly w.r.t. $t \geq 0$, and as $t \to +\infty$,

$$\|\chi(t)\|_m = O(t^{-d(\frac{1}{2} - \frac{1}{m})}).$$

Moreover, there exist vectors $\sigma_{+j} = (\beta_{+j}, E_{+j}, b_{+j}, v_{+j})$, such that as $t \to +\infty$,

$$|\sigma_j(t) - \sigma_{+j}(t)| = O(t^{-\delta}),$$

for some $\delta > 0$. Here $\sigma_{+j}(t)$ is the trajectory of (1.2) *with the initial data $\sigma_{+j}(0) = \sigma_{+j}$.*

2. Proof of the theorem

Up to some technical modifications the main line of the proof repeats that of [23].

2.1. Splitting of the motions. Following [23] we decompose the solution ψ as follows.

$$\psi(x, t) = \sum_{j=1}^{N} w(x, \sigma_j(t)) + \chi(x, t). \tag{2.1}$$

Here $\sigma_j(t) = (\beta_j(t), E_j(t), b_j(t), v_j(t))$ is an arbitrary trajectory in the set of admissible values of parameters, it is not a solution of (1.2) in general.

We fix the decomposition (2.1) by imposing the orthogonality conditions

$$\left\langle \vec{f}_j(t), \sigma_3 \vec{\xi}_k(E_j(t)) \right\rangle = 0, \quad j = 1, \dots, N, \;\; k = 0, \dots, 2d+1. \tag{2.2}$$

Here

$$\vec{f}_j = \begin{pmatrix} f_j \\ \bar{f}_j \end{pmatrix}, \quad \chi(x, t) = \exp(i\Phi_j) f_j(y_j, t),$$

$$\Phi_j = \beta_j(t) + v_j \cdot x/2, \quad y_j = x - b_j(t),$$

$< \cdot, \cdot >$ is the inner product in $L_2(\mathbb{R}^d \to \mathbb{C}^2)$.

Geometrically these conditions mean that for each t the vector $\vec{f}_j(t)$ belongs to the subspace of the continuous spectrum of the operator $L(E_j(t))$.

For ψ of the form (1.4) with $\min\limits_{\substack{j,k \\ j \neq k}}(|v_{jk}^0| + |b_{jk}^0|)$ sufficiently large, and with χ_0 sufficiently small in some L_p norm, the solvability of (2.2) is guaranteed by the non-degeneration of the corresponding Jacobi matrix, see lemma A1.1. So, one can assume that the initial decomposition (1.4) obeys (2.2). To prove the existence of a decomposition (2.1), (2.2) for $t > 0$, one can invoke a standard continuity type argument, see appendix 1 for the details.

Rewriting (2.1) as an equation for χ one gets

$$i\vec{\chi}_t = H(\vec{\sigma}(t))\vec{\chi} + N, \tag{2.3}$$

Here

$$\vec{\chi} = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}, \quad \vec{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^{(2d+2)N},$$

$$H(\vec{\sigma}) = -\triangle\sigma_3 + \sum_{j=1}^{N} \mathcal{V}(w_j),$$

$$\mathcal{V}(w) = (F(|w|^2) + F'(|w|^2)|w|^2)\sigma_3 + F'(|w|^2)\begin{pmatrix} 0 & w^2 \\ -\bar{w}^2 & 0 \end{pmatrix}, \quad w_j = w(x, \sigma_j).$$

The nonlinearity N is given by the following expression

$$N = N_0 + \sum_{j=1}^{N} e^{i\sigma_3\Phi_j} l(\sigma_j)\vec{\xi}_0(y_j, E_j),$$

$$N_0 = F(|\psi_s + \chi|^2)\begin{pmatrix} \psi_s + \chi \\ -\bar{\psi}_s - \bar{\chi} \end{pmatrix} -$$

$$\sum_{j=1}^{N}\left(F(|w_j|^2)\begin{pmatrix} w_j \\ -\bar{w}_j \end{pmatrix} + \mathcal{V}(w_j)\vec{\chi}\right), \quad \psi_s = \sum_{j=1}^{N} w_j,$$

$$l(\sigma_j) = \gamma_j' + \frac{1}{2}v_j' \cdot y_j + ic_j' \cdot \nabla\sigma_3 - iE_j'\partial_E\sigma_3,$$

where γ_j, c_j are defined as follows.

$$\beta_j(t) = \int_0^t ds(E_j(s) - \frac{|v_j(s)|^2}{4} - \frac{v_j'(s) \cdot b_j(s)}{2}) + \gamma_j(t), \quad b_j(t) = \int_0^t ds v_j(s) + c_j(t).$$

In terms of parameters (γ, E, c, v) (1.2) takes the form

$$\gamma' = 0, \; E' = 0, \; c' = 0, \; v' = 0.$$

Substituting the expression for χ_t from (2.3) into the derivative of the orthogonality conditions, one gets for $j = 1, \ldots, N$

$$ie(E_j)E_j' = \left\langle N_j, \sigma_3 e^{i\Phi_j}\vec{\xi}_0(\cdot - b_j, E_j)\right\rangle + \left\langle \vec{f}_j, l(\sigma_j)\vec{\xi}_0(E_j)\right\rangle,$$

$$n(E_j)v_j' = \left(\left\langle N_j, \sigma_3 e^{i\Phi_j}\vec{\xi}_k(\cdot - b_j, E_j)\right\rangle + \left\langle \vec{f}_j, l(\sigma_j)\vec{\xi}_k(E_j)\right\rangle\right)_{k=1,\ldots,d},$$

$$e(E_j)\gamma_j' = \left\langle N_j, \sigma_3 e^{i\Phi_j}\vec{\xi}_{d+1}(\cdot - b_j, E_j)\right\rangle + \left\langle \vec{f}_j, l(\sigma_j)\vec{\xi}_{d+1}(E_j)\right\rangle, \tag{2.4}$$

$$in(E_j)c_j' = -\left(\left\langle N_j, \sigma_3 e^{i\Phi_j}\vec{\xi}_{d+1+k}(\cdot - b_j, E_j)\right\rangle + \left\langle \vec{f}_j, l(\sigma_j)\vec{\xi}_{d+1+k}(E_j)\right\rangle\right)_{k=1,\ldots,d}.$$

Here

$$N_j = N_0 + \sum_{k, k\neq j} \mathcal{V}(w_k)\vec{\chi} + \sum_{k, k\neq j} e^{i\sigma_3\Phi_k} l(\sigma_k)\vec{\xi}_0(y_k, E_k), \quad j = 1, \dots, N,$$

$$e = \frac{d}{dE}\|\varphi\|_2^2, \ n = \frac{1}{2}\|\varphi\|_2^2.$$

The right hand side of (2.4) also contain the derivative $\vec{\sigma}'$, which enters linearly in $l(\sigma_k)$. In principle, system (2.4) can be solved with respect to derivative and together with equation (2.3) constitutes a complete system for $\vec{\sigma}$ and χ:

$$i\vec{\chi}_t = H(\vec{\sigma}(t))\vec{\chi} + N(\vec{\sigma}, \vec{\chi}), \tag{2.5}$$

$$\vec{\sigma}' = G(\vec{\sigma}, \vec{\chi}), \quad \chi|_{t=0} = \chi_0, \quad \sigma_j(0) = \sigma_{0j}. \tag{2.6}$$

2.2. Integral representations for χ. In this subsection we follow closely the constructions of Hagedorn [15] (developed in order to prove the asymptotic completeness for the charge transfer model), see also [21]. We start by rewriting (2.5) as an integral equation

$$\vec{\chi}(t) = \mathcal{U}_0(t, 0)\chi_0 - i\int_0^t \mathcal{U}_0(t, s)\left[\sum_{j=1}^N \mathcal{V}_j(s)\vec{\chi}(s) + N\right] ds, \tag{2.7}$$

Here $\mathcal{U}_0(t, \tau) = e^{i(t-\tau)\triangle\sigma_3}$, $\mathcal{V}_j = \mathcal{V}(w_j)$.

Next we introduce the one soliton adiabatic propagators $\mathcal{U}_j^A(t, \tau)$:

$$i\mathcal{U}_{j\,t}^A(t, \tau) = L_j(t)\mathcal{U}_j^A(t, \tau), \quad \mathcal{U}_j^A(t, \tau)|_{t=\tau} = I,$$

$$L_j(t) = -\triangle\sigma_3 + \tilde{\mathcal{V}}_j(t) + R_j(t), \quad R_j(t) = iT_{0j}(t)[P_j'(t), P_j(t)]T_{0j}^*(t),$$

$$\tilde{\mathcal{V}}_j(t) = T_{0j}(t)T_j(t)V(E_{0j})T_j^*(t)T_{0j}^*(t), \quad P_j(t) = T_j(t)\hat{P}(E_{0j})T_j^*(t).$$

Here

$$T_{0j}(t) = B_{\beta_{0j}(t), b_{0j}(t), v_{0j}}, \quad T_j(t) = B_{\theta_j(t), a_j(t), 0},$$

$$\theta_j = \int_0^t ds\left(E_j(s) - E_{0j} + \frac{|v_j(s) - v_{0j}|^2}{4}\right), \quad a_j = \int_0^t ds(v_j(s) - v_{0j}),$$

$$(B_{\beta, b, v}f)(x) = e^{i\beta\sigma_3 + i\frac{v\cdot x}{2}\sigma_3}f(x - b),$$

$\sigma_{0j}(t) = (\beta_{0j}(t), E_{0j}, b_{0j}(t), v_{0j})$ being the solution of (1.2) with initial data $\sigma_{0j}(0) = \sigma_{0j}$. Obviously,

$$P_j^A(t)\mathcal{U}_j^A(t, \tau) = \mathcal{U}_j^A(t, \tau)P_j^A(\tau),$$

where

$$P_j^A(t) = T_{0j}(t)P_j(t)T_{0j}^*(t).$$

Write the solution χ as the sum:

$$\vec{\chi}(t) = \vec{h}_j(t) + \vec{k}_j(t), \quad \vec{h}_j(t) = P_j^A(t)\vec{\chi}(t).$$

Using the adiabatic evolution $\mathcal{U}_j^A(t,\tau)$ one can write the following representation for $h_j(t)$

$$\vec{h}_j(t) = \mathcal{U}_j^A(t,0)P_j^A(0)\vec{\chi}_0 - i\int_0^t \mathcal{U}_j^A(t,s)P_j^A(s)[\sum_{m,\, m\neq j} \mathcal{V}_m(s)\vec{\chi}(s) + D_j(s)]ds, \tag{2.8}$$

Here

$$D_j = N + (\mathcal{V}_j - \tilde{\mathcal{V}}_j)\vec{\chi} - R_j\vec{\chi}. \tag{2.9}$$

Combining (2.7), (2.8) one gets finally

$$\vec{\chi} = \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}, \tag{2.10}$$

where

$$\text{(I)} = \mathcal{U}_0(t,0)\vec{\chi}_0 - i\sum_j \int_0^t ds\mathcal{U}_0(t,s)\tilde{\mathcal{V}}_j(s)\mathcal{U}_j^A(s,0)P_j^A(0)\vec{\chi}_0,$$

$$\text{(II)} = -\sum_{\substack{j,m\\ j\neq m}} \int_0^t dsK_j(t,s)\mathcal{V}_m(s)\vec{\chi}(s),$$

$$\text{(III)} = -i\int_0^t ds\mathcal{U}_0(t,s)D,$$

$$\text{(IV)} = -\sum_j \int_0^t dsK_j(t,s)D_j(s).$$

Here

$$D = N + \sum_j \left(\tilde{\mathcal{V}}_j\vec{k}_j + (\mathcal{V}_j - \tilde{\mathcal{V}}_j)\vec{\chi}\right), \tag{2.11}$$

$$K_j(t,s) = \int_s^t d\rho\mathcal{U}_0(t,\rho)\tilde{\mathcal{V}}_j(\rho)\mathcal{U}_j^A(\rho,s)P_j^A(s).$$

The relations (2.4), (2.7), (2.10) make up the final form of the equation which is used to prove theorem 1.1.

2.3. Estimates of solitons parameters. Following [4, 23] we consider (2.4), (2.7), (2.10) on some finite interval $[0,t_1]$ and then study the limit $t_1 \to +\infty$. On the interval $[0,t_1]$ we introduce a natural system of norms for the components of the solution ψ:

$$M_0(t) = \sum_{j=1}^N |\gamma_j(t) - \beta_{0j}| + |E_j(t) - E_{j0}| + |c_j(t) - b_{0j}| + |v_j(t) - v_{0j}|,$$

$$M_1(t) = \sum_{j=1}^N \| < y_j >^{-\nu} \chi(t)\|_2, \quad M_2(t) = \|\chi(t)\|_{2p+2}, \quad \nu > \frac{d+2}{2},$$

without loss of generality one can assume that $m = 2p+2$.

These norms generate the system of majorants

$$\mathbb{M}_0(t) = \sup_{0\le\tau\le t} M_0(\tau),\ \mathbb{M}_l(t) = \sup_{0\le\tau\le t} M_l(\tau)\rho^{-\mu_l}(\tau),\ l=1,2,\quad \hat{\mathbb{M}}_k = \mathbb{M}_k(t_1).$$

Here $1<\mu_1<\frac{3}{2}$ if $d=3$ and $1<\mu_1=\frac{dp}{2}$ for $d\ge 4$, $\mu_2 = d(\frac{1}{2}-\frac{1}{2p+2})$,

$$\rho(t) = <t>^{-1} + \sum_{\substack{j,k\\ j\neq k}} <t-t_{jk}>^{-1},$$

t_{jk} being "the collision times" that are defined as follows. We set $t_{jk}=0$ if $t^0_{jk}\le 0$. For (j,k) such that $t^0_{jk}>0$, we define t_{jk} by the relation,

$$\int_0^{t_{jk}} ds \frac{\tilde v_{jk}(s)\cdot v^0_{jk}}{|v^0_{jk}|^2} = t^0_{jk},$$

where

$$\tilde v_{jk}(t) = \begin{cases} v_{jk}(t), \text{ if } t\le t_1, \\ v_{jk}(t_1), \text{ if } t>t_1, \end{cases} \quad v_{jk}(t) = v_j(t)-v_k(t).$$

Let us mention that

(i) t_{jk} are well defined provided $|v_{jk}(t)-v^0_{jk}|<v_0$, $0\le t\le t_1$;

(ii) the collision times t_{jk} belonging to the interval $[0,t_1]$ "do not depend on t_1".

It follows directly from the definition of M_0 that

$$|\theta'_j(t)|,\ |a'_j(t)| \le M_0(t)+M_0^2(t),\ \ |b_j(t)-\tilde b_j(t)|\le M_0(t), \tag{2.12}$$

$$|\Phi_j(x,t)-\tilde\Phi_j(x,t)| \le M_0(t) <x-b_j(t)> + \mathbb{M}_0(t)\int_0^t ds |c'_j(s)|, \tag{2.13}$$

where

$$\tilde b_j(t) = b_{0j}(t)+a_j(t),\quad \tilde\Phi_j(x,t) = \beta_{0j}(t)+\theta_j(t)+v_{0j}\cdot x/2.$$

It is also easy to check that $\tilde b_{jk} = \tilde b_j - \tilde b_k$ admits the estimates

$$|\tilde b_{jk}(t)| \ge c|v^0_{jk}||t-t_{jk}|, \tag{2.14}$$

$$|\tilde b_{jk}(t)| \ge c(\min_{s\ge 0}|b^0_{jk}(s)| + |v^0_{jk}||t-t_{jk}|) - c,\quad t^0_{jk}\le \kappa <r^0_{jk}> \tag{2.15}$$

provided $M_0(t)\le c$ for $0\le t\le t_1$. Here and below c is used as a general notation of positive constants that depend only on v_0,κ and eventually on $E_j, j=1,\dots,N$, in that case they can be chosen uniformly with respect to E_j in some finite vicinity of E_{0j}.

Consider relations (2.4). Since

$$|N_0| \le c\left(\sum_{\substack{j,k\\ j\neq k}} |w_j||w_k|(1+|\chi|) + \begin{cases} |\chi|^2+|\chi|^{2p+1} & \text{if } p>\frac{1}{2}, \\ |\chi|^2 & \text{if } p\le \frac{1}{2} \end{cases}\right)$$

and

$$|<e^{i\sigma_3\Phi_j}l(\sigma_j)\vec{\xi}_0(\cdot-b_j,E_j),\sigma_3e^{i\sigma_3\Phi_k}\vec{\xi}_l(\cdot-b_k,E_k)>|=O(|\lambda_j|e^{-c|b_{jk}|}|v_{jk}|^{-\infty}),$$

$j\neq k$, $\lambda_j=(\gamma_j',E_j',c_j',v_j')$, $b_{jk}=b_j-b_k$, one gets immediately from (2.4)

$$|\lambda_j(t)|\leq W(\mathbb{M})[\sum_{\substack{i,l\\ i\neq k}}e^{-c|b_{ik}(t)|}+\left(\mathbb{M}_1^2(t)+\mathbb{M}_2^2(t)\right)\rho^{2\mu_1}(t)]. \tag{2.16}$$

We use $W(\mathbb{M})$ as a general notation for functions of $\mathbb{M}_0$, $\mathbb{M}_1$, $\mathbb{M}_2$, which are bounded in some finite vicinity of the point $\mathbb{M}_l=0$, $l=0,1,2$, and may acquire $+\infty$ out some larger vicinity. They depend only on v_0, κ_0, E_{j0}, $j=1,\dots,N$ and can be chosen to be spherically symmetric and monotone. In all the formulas where W appear it would not be hard to replace them by some explicit expressions but such expressions are useless for our aims.

Combining (2.13), (2.16) one gets

$$|\Phi_j(x,t)-\tilde{\Phi}_j(x,t)|\leq W(\mathbb{M})\mathbb{M}_0(t)<x-b_j(t)>. \tag{2.17}$$

Integrating (2.16) and taking into account (2.14), (2.15) we obtain

$$\mathbb{M}_0\leq W(\hat{\mathbb{M}})[\epsilon+\mathbb{M}_1^2+\mathbb{M}_2^2]. \tag{2.18}$$

Consider the vectors $\vec{k}_j(t)=(I-P_j^A(t))\vec{\chi}(t)$, $\vec{k}_j(x,t)=\sum_{l=0}^{2d+1}k_{jl}(t)e^{i\tilde{\Phi}_j\sigma_3}\vec{\xi}_l(x-\tilde{b}_j(t),E_{0j})$. The orthogonality conditions (2.2) together with (2.12), (2.17) lead immediately to the estimate:

$$|k_{jl}(t)|\leq W(\mathbb{M})\mathbb{M}_0(t)\|e^{-c|x-b_j(t)|}\chi(t)\|_2\leq W(\mathbb{M})\mathbb{M}_0(t)\mathbb{M}_1(t)\rho^{\mu_1}(t). \tag{2.19}$$

2.4. Linear estimates. To study the behavior of solutions of the integral equation (2.10) we need some estimates of the evolution operators $\mathcal{U}_m^A(t,\tau)P_m^A(\tau)$. The necessary estimates are collected in this subsection, the proofs being removed to the appendices.

Lemma 2.1. *For any $x_0,x_1\in\mathbb{R}^d$, $0\leq\tau\leq t\leq t_1$,*

$$\|\langle x-x_0\rangle^{-\nu_0}\mathcal{U}_j^A(t,\tau)P_j^A(\tau)f\|_2\leq W(\hat{\mathbb{M}})\langle t-\tau\rangle^{-d/2}\|\langle x-x_1\rangle^{\nu_0}f\|_2. \tag{2.20}$$

The function W here is independent of x_0,x_1 and t_1.

See appendix 2 for the proof.

Remark. Due to the representation

$$\mathcal{U}_j^A(t,\tau)P_j^A(\tau)f=P_j^A(t)\mathcal{U}_0(t,\tau)f-i\int_\tau^t ds\,\mathcal{U}_j^A(t,s)P_j^A(s)(\tilde{\mathcal{V}}_j(s)+R_j(s))\mathcal{U}_0(s,\tau)f,$$

and the estimate

$$|(R_j(t)f)(x)|\leq W(\mathbb{M})e^{-c|x-b_j(t)|}(|\theta_j'|+|a_j'|)\|e^{-c|x-b_j(t)|}f\|_2$$

$$\leq W(\mathbb{M}) e^{-c|x-b_j(t)|} \mathbb{M}_0(t) \| e^{-c|x-b_j(t)|} f\|_2, \tag{2.21}$$

(2.20) leads immediately to the inequality

$$\| \langle x-x_0\rangle^{-\nu_0} \mathcal{U}_j^A(t,\tau) P_j^A(\tau) f\|_2 \leq W(\hat{\mathbb{M}}) \frac{(\|f\|_{p_1'} + \|f\|_{p_2'})}{|t-\tau|^{d(\frac{1}{2}-\frac{1}{p_1})} \langle t-\tau\rangle^{d(\frac{1}{p_1}-\frac{1}{p_2})}}, \tag{2.22}$$

where $2 \leq p_1 < \frac{2d}{d-2} < p_2 \leq \infty$, $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $i = 1, 2$. Obviously, the same estimate is valid for $K_j(t,\tau)$:

$$\| \langle x-x_0\rangle^{-\nu_0} K_j(t,\tau) f\|_2 \leq W(\hat{\mathbb{M}}) \frac{(\|f\|_{p_1'} + \|f\|_{p_2'})}{|t-\tau|^{d(\frac{1}{2}-\frac{1}{p_1})} \langle t-\tau\rangle^{-d(\frac{1}{p_1}-\frac{1}{p_2})}}. \tag{2.23}$$

The key point of our analysis is the following lemma that is essentially lemma 3.6 of [15].

Lemma 2.2. *Introduce the operators $T_{jki}(t,\tau)$, $j,k,i = 1,\ldots,N$, $i \neq k$*

$$T_{jki}(t,\tau) = A_j(t) K_k(t,\tau) A_i(\tau),$$

where $A_j(t)$ is the multiplication by $< x - b_j(t) >^{-\nu}$. Then, for $0 \leq t \leq t_1$

$$\int_0^t d\tau \|T_{jki}(t,\tau)\| \leq W(\hat{\mathbb{M}})(\epsilon_{ik}^{\nu_1} + \mathbb{M}_0(t)),$$

with some $\nu_1 > 0$. The norm $\|\cdot\|$ here stands for the $L_2 \to L_2$ operator norm.

See appendix 3 for the proof.

2.5. Estimates of the nonlinear terms. Here we derive the necessary estimates of D, D_j. We write D as the sum:

$$D = D^0 + D^1 + D^2,$$

where

$$D^0 = N_{00} + \sum_j \left((\mathcal{V}_j - \tilde{\mathcal{V}}_j)\vec{\chi} + \tilde{\mathcal{V}}_j \vec{k}_j + e^{i\Phi_j\sigma_3} l(\sigma_j) \vec{\xi}_0(\cdot - b_j, E_j) \right),$$

$$N_{00} = F(|\psi_s|^2)\begin{pmatrix} \psi_s \\ -\bar{\psi}_s \end{pmatrix} - \sum_j F(|w_j|^2)\begin{pmatrix} w_j \\ -\bar{w}_j \end{pmatrix} + \mathcal{V}(\psi_s)\vec{\chi} - \sum_j \mathcal{V}_j \vec{\chi},$$

$$D^1 = F(|\psi_s+\chi|^2)\begin{pmatrix} \psi_s + \chi \\ -\bar{\psi}_s - \bar{\chi} \end{pmatrix} - F(|\psi_s|^2)\begin{pmatrix} \psi_s \\ -\bar{\psi}_s \end{pmatrix} - \mathcal{V}(\psi_s)\vec{\chi} - F(|\chi|^2)\begin{pmatrix} \chi \\ -\bar{\chi} \end{pmatrix},$$

$$D^2 = F(|\chi|^2)\begin{pmatrix} \chi \\ -\bar{\chi} \end{pmatrix}.$$

In a similar way,

$$D_j = D_j^0 + D^1 + D^2, \quad j = 1, \ldots N,$$

where

$$D_j^0 = N_{00} + (\mathcal{V}_j - \tilde{\mathcal{V}}_j)\vec{\chi} - R_j\vec{\chi} + \sum_k e^{i\Phi_k\sigma_3} l(\sigma_k)\vec{\xi}_0(\cdot - b_k, E_k).$$

Estimating N_{00} by

$$|N_{00}| \le c(1+|\chi|)\sum_{\substack{j,k\\ k\neq j}} |w_j||w_k|, \tag{2.24}$$

and using (2.12), (2.17), (2.19), (2.21) one gets

$$|D^0|, |D_j^0| \le W(\mathbb{M})[(1+|\chi|)\sum_{\substack{i,k\\ i\neq k}} e^{-c(|x-b_i|+|x-b_k|)}$$

$$+\sum_i e^{-c|x-b_i|}(|\lambda_i| + \mathbb{M}_0(t)|\chi| + \mathbb{M}_0(t)M_1(t))].$$

which together with (2.16) leads to the inequality

$$\|D^0\|_{L_1\cap L_2}, \|D_j^0\|_{L_1\cap L_2} \le W(\mathbb{M})[e^{-c|b_{jk}(t)|} + (\mathbb{M}_0\mathbb{M}_1 + \mathbb{M}_1^2 + \mathbb{M}_2^2)\rho^{\mu_1}(t)]. \tag{2.25}$$

Consider D^1, D^2. We estimate them as follows.

$$|D^1 + D^2| \le W(\mathbb{M})[|\psi_s||\chi|^2 + |\chi|^3 + |\chi|^{2p+1}], \quad \text{if } d=3,$$

$$|D^1| \le W(\mathbb{M})|\psi_s||\chi|^2,\ |D^2| \le |\chi|^{2p+1}, \quad \text{if } \frac{1}{2} < p < 1, \tag{2.26}$$

$$|D^1 + D^2| \le W(\mathbb{M})|\chi|^{2p+1}, \quad \text{if } p \le \frac{1}{2}.$$

These inequalities imply for $r' = \frac{2}{1+p}$,

$$\|D^1 + D^2\|_{L_1\cap L_{m'}} \le W(\mathbb{M})[\mathbb{M}_1^2 + \mathbb{M}_1^{2-\frac{1}{p}}\mathbb{M}_2^{\frac{1}{p}} + \mathbb{M}_2^{1+\frac{1}{p}}]\rho^{\mu_1}(t), \quad \text{if } d=3,$$

(2.27)

$$\|D^1\|_{L_1\cap L_{m'}} + \|D^2\|_{L_{r'}\cap L_{m'}} \le W(\mathbb{M})[\mathbb{M}_1^2 + \mathbb{M}_1^{2-\frac{1}{p}}\mathbb{M}_2^{\frac{1}{p}} + \mathbb{M}_2^{1+p}]\rho^{\mu_1}(t), \text{ if } \frac{1}{2} < p < 1$$

$$\|D^1 + D^2\|_{L_{r'}\cap L_{m'}} \le W(\mathbb{M})\mathbb{M}_2^{1+p}\rho^{\mu_1}(t), \quad \text{if } p \le \frac{1}{2}.$$

2.6. Estimates of χ in $L_{2,loc}$. To estimate $M_1(t)$ we use representation (2.10). By (2.22), for the first term (I) one has

$$\| < y_j >^{-\nu} \text{(I)}\|_2 \le W(\mathbb{M})\mathcal{N} < t >^{-d/2}. \tag{2.28}$$

Consider expression (II):

$$\| < y_j >^{-\nu} \text{(II)}\|_2 \le W(\mathbb{M})\mathbb{M}_1(t)\sum_{\substack{k,i\\ k\neq i}}\int_0^t ds\|T_{jki}(t,s)\|\rho^{\mu_1}(s). \tag{2.29}$$

By lemma 2.1,

$$\|T_{jki}(t,s)\| \leq W(\mathbb{M}) < t-s >^{-d/2} .$$

So, the integral in the right hand side of (2.29) can be estimated as follows.

$$\int_0^t ds \|T_{jki}(t,s)\| \rho^{\mu_1}(s) \leq \left(\int_0^t ds \|T_{jki}(t,s)\| \rho^{d/2}(s) \right)^{\frac{2\mu_1}{d}} \left(\int_0^t ds \|T_{jki}(t,s)\| \right)^{1-\frac{2\mu_1}{d}}$$

$$\leq W(\mathbb{M})(\mathbb{M}_0^\theta + \epsilon_{ik}^{\nu_2}) \left(\int_0^t ds < t-s >^{-d/2} \rho^{d/2}(s) \right)^{\frac{2\mu_1}{d}}$$

$$\leq W(\mathbb{M})(\mathbb{M}_0^\theta + \epsilon_{ik}^{\nu_2}) \rho^{\mu_1}(t),$$

$0 < \theta = 1 - \frac{2\mu_1}{d}$, $\nu_2 = \theta\nu_1$. At the second step here we have used lemma 2.2. Thus,

$$\| < y_j >^{-\nu} \text{(II)} \|_2 \leq W(\hat{\mathbb{M}})(\mathbb{M}_0^\theta + \epsilon_{ik}^{\nu_2}) \mathbb{M}_1(t) \rho^{\mu_1}(t). \tag{2.30}$$

Consider the two last terms in the r.h.s. of (2.10). By (2.25), (2.23) (with $p_1 = 2$, $p_2 = \infty$) one has

$$\| < y_j >^{-\nu} \mathcal{U}_0(t,s) D^0(s) \|_2, \ \| < y_j >^{-\nu} K_m(t,s) D_m^0(s) \|_2 \leq W(\mathbb{M}) < t-s >^{-d/2}$$

$$\times [\sum_{\substack{i,k \\ i \neq k}} e^{-|b_{ik}(s)|} + (\mathbb{M}_0(t)\mathbb{M}_1(t) + \mathbb{M}_1^2(t) + \mathbb{M}_2^2(t)) \rho^{\mu_1}(s)]. \tag{2.31}$$

Using (2.27), (2.23) one can estimate the contribution of D^1, D^2 as follows.

$$\| < y_j >^{-\nu} \mathcal{U}_0(t,s)(D^1(s) + D^2(s)) \|_2, \ \| < y_j >^{-\nu} K_m(t,s)(D^1(s) + D^2(s)) \|_2$$

$$\leq W(\mathbb{M})[\mathbb{M}_1^2(t) + \mathbb{M}_2^{r_1}(t)] |t-s|^{-\mu_2} < t-s >^{-\mu_1+\mu_2} \rho^{\mu_1}(s). \tag{2.32}$$

Here $1 < r_1 = 1 + \min\{p, p^{-1}\} < 2$. Combining (2.31), (2.32) and integrating with respect to s one gets

$$\| < y_j >^{-\nu} \text{(III)} \|_2, \| < y_j >^{-\nu} \text{(IV)} \|_2 \leq W(\mathbb{M}) [\sum_{\substack{i,k \\ i \neq k}} \int_0^t ds \frac{e^{-|b_{ik}(s)|}}{< t-s >^{d/2}}$$

$$+ (\mathbb{M}_0\mathbb{M}_1 + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1}) \rho^{\mu_1}(t)],$$

or taking into account (2.14), (2.15),

$$\| < y_j >^{-\nu} \text{(III)} \|_2, \| < y_j >^{-\nu} \text{(IV)} \|_2 \leq W(\hat{\mathbb{M}}) [\epsilon + \mathbb{M}_0\mathbb{M}_1 + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1}] \rho^{\mu_1}(t). \tag{2.33}$$

Combining (2.28), (2.30), (2.33), one obtains

$$\mathbb{M}_1 \leq W(\hat{\mathbb{M}})[\mathcal{N} + \epsilon^{\nu_2} + \mathbb{M}_0^\theta \mathbb{M}_1 + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1}].$$

Changing if necessary the coefficient function W one can simplify this inequality:

$$\mathbb{M}_1 \leq W(\hat{\mathbb{M}})[\mathcal{N} + \epsilon^{\nu_2} + \mathbb{M}_2^{r_1}]. \tag{2.34}$$

2.7. Closing of the estimates. Here we derive a L_m estimate of χ which will close the system of the inequalities for the majorants. To estimate L_m - norm of χ we use representation (2.7). By (2.24), (2.27),

$$\|N\|_{m'} \le W(\mathbb{M})[\sum_{\substack{k,i \\ k\neq i}} e^{-c|b_{ik}(t)|} + \mathbb{M}_1^2 + \mathbb{M}_2^{r_1})\rho^{\mu_1}(t)].$$

As a consequence,

$$\mathbb{M}_2 \le W(\hat{\mathbb{M}})[\mathcal{N} + \epsilon^{1-\mu_2} + \mathbb{M}_1]. \tag{2.35}$$

Here we have made use of the inequality

$$\int_0^t ds \frac{e^{-c|b_{ik}(s)|}}{|t-s|^{\mu_2}} \le W(\hat{\mathbb{M}}) \frac{\epsilon_{ik}^{1-\mu_2}}{< t - t_{ik} >^{\mu_2}},$$

which is an immediate consequence of (2.14), (2.15).

Combining (2.18), (2.34), (2.35) one gets

$$\hat{\mathbb{M}}_1,\, \hat{\mathbb{M}}_2 \le W(\hat{\mathbb{M}})(\mathcal{N} + \epsilon^{\nu_3}), \quad \hat{\mathbb{M}}_0 \le W(\hat{\mathbb{M}})(\mathcal{N}^2 + \epsilon^{2\nu_3}), \tag{2.36}$$

$\nu_3 = \min\{\frac{1}{2}, \nu_2, 1-\mu_2\} > 0$, the coefficient functions $W(\mathbb{M})$ being independent of t_1. These inequalities mean that for $\mathcal{N}$ and ϵ sufficiently small $\mathbb{M}$ can belong either to a small neighborhood of zero or to some domain whose distance from zero is bounded from below uniformly with respect to $\mathcal{N}$, ϵ. Since $\hat{\mathbb{M}}_l$ are continuous functions of t_1 and for $t_1 = 0$ are small only the first possibility can be realized. This means that for $\mathcal{N}$ and ϵ in some finite vicinity of zero,

$$\mathbb{M}_1(t),\, \mathbb{M}_2(t) \le c(\mathcal{N} + \epsilon^{\nu_3}), \quad \mathbb{M}_0(t) \le c(\mathcal{N}^2 + \epsilon^{2\nu_3}), \quad 0 \le t \le t_1.$$

The constant c here is independent of $\mathcal{N}$, ϵ, t_1. Since t_1 is arbitrary these estimates are valid, in fact, for all $t \ge 0$. More precisely, one has

(2.37)
$$M_0(t) \le c(\mathcal{N}^2 + \epsilon^{2\nu_3}), \;\; M_1(t) \le c(\mathcal{N} + \epsilon^{\nu_3})\rho_\infty^{\mu_1}(t), \;\; M_2(t) \le c(\mathcal{N} + \epsilon^{\nu_3})\rho_\infty^{\mu_2}(t),$$

where $\rho_\infty(t)$ is the weight function corresponding to $t_1 = \infty$:

$$\rho_\infty(t) = < t >^{-1} + \sum_{\substack{j,k \\ j\neq k}} < t - t_{jk}^\infty >^{-1},$$

$t_{jk}^\infty = 0$ if $t_{jk}^0 \le 0$, and

$$\int_0^{t_{jk}^\infty} ds \frac{v_{jk}(s)\cdot v_{jk}^0}{|v_{jk}^0|^2} = t_{jk}^0,$$

if $t_{jk}^0 > 0$.

By (2.15), (2.16), the estimates (2.36) imply the existence of the limit trajectories $\sigma_{+j}(t) = (\beta_{+j}(t), E_{+j}, b_{+j}(t), v_{+j})$, $j = 1, \ldots, N$,

$$b_{+j}(t) = v_{+j}t + b_{+j}, \quad v_{+j} = v_{0j} + \int_0^\infty ds v_j'(s),$$

$$b_{+j} = b_{0j} + \int_0^\infty ds(c_j'(s) + v_j(s) - v_{+j}),$$

$$\beta_{+j}(t) = (E_{+j} - \frac{|v_{+j}|^2}{4})t + \beta_{+j}, \quad E_{+j} = E_{0j} + \int_0^\infty dsE_j'(s),$$

$$\beta_{+j} = \beta_{0j} + \int_0^\infty ds\big(E_j - E_{+j} + \frac{|v_j - v_{+j}|^2}{4} + \gamma_j' - \frac{1}{2}v_j' \cdot c_j\big).$$

Obviously, as $t \to +\infty$,

$$|E_j(t) - E_{+j}|, \, |v_j(t) - v_{+j}| = 0(t^{-2\mu_1+1}),$$

$$|b_j(t) - b_{+j}(t)|, \, |\beta_j(t) - \beta_{+j}(t)| = O(t^{-2\mu_1+2}).$$

Appendix 1

Here we outline the arguments needed for the proof of the existence of a decomposition (2.1) satisfying (2.2) for all $t \geq 0$. We begin with the following lemma. Given N solitons $w(\sigma_{0j})$, $\sigma_{0j} = (\beta_{0j}, E_{0j}, b_{0j}, v_{0j})$, $j = 1, \dots, N$, we define the effective coupling parameter $\delta(\vec{\sigma}_0)$, $\vec{\sigma}_0 = (\sigma_{01}, \dots, \sigma_{0N})$,

$$\delta(\vec{\sigma}_0) = \max_{j \neq k}(|v_{jk}^0| + |b_{jk}^0|)^{-1}.$$

For $\chi \in L_p(\mathbb{R}^d)$, $\vec{\sigma} = (\sigma_1, \dots, \sigma_N)$, $\sigma_j = (\beta_j, E_j, b_j, v_j) \in \mathbb{R} \times \mathcal{A} \times \mathbb{R}^d \times \mathbb{R}^d$, $j = 1, \dots, N$, consider the functionals $F_{j,l}(\vec{\sigma}, \chi; \vec{\sigma}_0)$, $j = 1, \dots, N$, $l = 0, \dots 2d+1$,

$$F_{j,l}(\vec{\sigma}, \chi; \vec{\sigma}_0) = \left\langle \vec{\chi} + \sum_{k=1}^N \vec{w}(\sigma_{0k}) - \vec{w}(\sigma_k), \sigma_3 \vec{\zeta}_l(\sigma_j) \right\rangle,$$

where

$$\vec{w} = \begin{pmatrix} w \\ \bar{w} \end{pmatrix}, \quad \vec{\zeta}_l(x, \sigma) = e^{i\beta\sigma_3 + i\frac{x \cdot v}{2}\sigma_3} \vec{\xi}_l(x - b, E), \;\; \sigma = (\beta, E, b, v).$$

Set $F_j = (F_{j,0}, \dots, F_{j,2d+1})$, $F = (F_1, \dots, F_N)$.

Lemma A1.1. *Let $E_{0j} \in \mathcal{A}_0$, $j = 1, \dots, N$. There exist constants $n_0 > 0$, $\delta_0 > 0$, $K > 0$, depending only on E_{0j}, $j = 1, \dots, N$ such that if $\delta(\vec{\sigma}_0) \leq \delta_0$ and $\|\chi\|_p \leq n_0$ then the equation*

$$F(\vec{\sigma}, \chi; \vec{\sigma}_0) = 0$$

has a unique solution $\vec{\sigma} = (\sigma_1, \dots, \sigma_N)$, $\vec{\sigma}$ being a C^1 function of χ, that satisfies

$$\text{(A1.1)} \quad |\beta_j - b_{0j} + \frac{1}{2}(v_j - v_{0j}) \cdot b_{0j}| + |E_j - E_{0j}| + |b_j - b_{0j}| + |v_j - v_{0j}| \leq K\|\chi\|_p.$$

Remark. It follows directly from (A1.1) that

(i) for some constant K_1

$$\|\chi + \sum_{k=1}^N w(\sigma_{0k}) - w(\sigma_k)\|_p \leq K_1 \|\chi\|_p,$$

(ii) if for some pair (j,k), $t^0_{jk} \le \kappa_0 < r^0_{jk} >$ then the new collision time $t_{jk} = -\frac{b_{jk}\cdot v_{jk}}{|v_{jk}|^2}$ satisfies a similar estimate with a constant $\kappa = \kappa_0(1 + O(\|\chi\|_p)$.

Proof of Lemma A1.1. Let us pass from $\vec\sigma$ to a new system of parameters $\vec\lambda = (\lambda_1,\dots,\lambda_N)$,

$$\lambda_j = (\beta_j - \beta_{0j} + \frac{1}{2}(v_j - v_{0j})\cdot b_{0j}, E_j, b_j - b_{0j}, v_j - v_{0j}).$$

We represent $F(\vec\sigma,\chi;\vec\sigma_0)$ as the sum

$$F = F^0 + F^1 + F^2,$$

$F^0_j = \Phi(\lambda_j, E_{0j})$, $\Phi = (\Phi_0,\dots\Phi_{2d+1})$,

$$\Phi_l(\lambda,E) = \left\langle \vec\xi_0(E) - \vec\zeta_0(\lambda), \vec\zeta_l(\lambda)\right\rangle,$$

$$F^1_{j,l} = \sum_{k,\,k\neq j}\left\langle \vec\zeta_0(\sigma_{0k}) - \vec\zeta_0(\sigma_k), \vec\zeta_l(\sigma_j)\right\rangle.$$

At last,

$$F^2_{j,l} = G_l(\lambda_j, f_j), \quad \chi(x) = e^{i\beta_{0j} + i\frac{v_{0j}\cdot x}{2}} f_j(x - b_{0j}),$$

$G_l(\lambda, f) =< \vec f, \sigma_3\vec\zeta_l(\lambda) >$ is a C^1 function of f and λ.

The direct calculations give

$$|\det \nabla_\lambda \Phi(\lambda,E)|\big|_{\lambda=(0,E,0,0)} = e^2(E)n^{2d}(E). \tag{A1.2}$$

Set $\vec\lambda_0 = (\lambda_{01},\dots,\lambda_{0N})$, $\quad \lambda_{0j} = (0, E_{0j}, 0, 0)$. By (A1.2),

$$|\det \nabla_{\vec\lambda} F^0|\big|_{\vec\lambda=\vec\lambda_0} = \prod_{j=1}^N e^2(E_{0j})n^{2d}(E_{0j}) \tag{A1.3}$$

is nonzero if $E_{0j}\in\mathcal{A}_0$, $j = 1,\dots,N$.

Consider F^1. It is not difficult to check that for $\vec\lambda$ in some finite vicinity of $\vec\lambda_0$ the derivative $\nabla_{\vec\lambda}F^1$ satisfies the inequality

$$|\nabla_{\vec\lambda}F^1| \le C\delta(\vec\sigma_0), \tag{A1.4}$$

constant C depending only on E_{0j}.

By the implicit function theorem, the desired result is a direct consequence of (A1.3), (A1.4). ■

To prove the existence of a decomposition (2.1) satisfying (2.2) for all $t > 0$ we use some standard continuity type arguments. Since $\psi \in C(\mathbb{R}\to H^1)$ there exists a small interval $[0,t_1]$ where the constructions of lemma A1.1 can be used. This leads to a representation (2.1) satisfying the orthogonality conditions for $t\in[0,t_1]$. For the components of such a representation estimates (2.15), (2.36) give

$$|E_{0j} - E| \le C(\mathcal{N}^2 + \epsilon^{2\nu_3}), \quad (|v_{jk}| + |b_{jk}|)^{-1} \le C\epsilon,$$

$$\|\chi(t)\|_m \le C(\mathcal{N} + \epsilon^{\nu_3}),$$

which allows us to extend decomposition (2.1), (2.2) on a larger interval $[0, t_1 + t_2]$ with some $t_2 > 0$. On this new interval the same estimates hold, so one can continue the procedure with steps of the same length t_2. As a result, one gets a decomposition (2.1) satisfying (2.2) for all $t\ge 0$.

Appendix 2

Here we prove lemma 2.1. Consider the equation

$$i\chi_t = \mathcal{L}(t)\chi, \quad \mathcal{L}(t) = (-\triangle + E)\sigma_3 + \mathcal{V}(t) + i[P'(t), P(t)], \tag{A2.1}$$

$$\mathcal{V}(t) = T(t)V(E)T^*(t), \quad P(t) = T(t)\hat{P}(E)T^*(t),$$

where $T(t) = B_{\theta(t),a(t),0}$. We denote the corresponding propagator by $U(t,\tau)$. Clearly,

$$\mathcal{U}_j^A(t,\tau) = B_{\sigma_{0j}(t)}U(t,\tau)\big|_{\theta=\theta_j,\, a=a_j, E=E_{0j}} B_{\sigma_{0j}(\tau)}.$$

We shall assume that for some positive constants n, R, δ_1,

$$|\theta'(t)| + |a'(t)| \leq n, \tag{A2.2}$$

$$|\theta''(t)| + |a''(t)| \leq \sum_{l=0}^{L} < R(t-t_l) >^{-2-\delta_1}, \tag{A2.3}$$

$t \in \mathbb{R}_+$. Here $L \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_L$. One has the following lemma.

Lemma A2.1. *For any $x_0, x_1 \in \mathbb{R}^d$, $t \geq 0$,$\tau \geq 0$,*

$$\| \langle x - b_0 \rangle^{-\nu_0} U(t,\tau)P(\tau)f\|_2 \leq C \langle t-\tau \rangle^{-d/2} \| \langle x - x_1 \rangle^{\nu_0} f\|_2,$$

provided n is sufficiently small and R is sufficiently large: $n + R^{-1} \leq C$.

In this appendix we use C as a general notation for constants that depend only on M, δ, E and can be chosen uniformly with respect to E in compact subsets of $\mathcal{A}_0$.

It follows from (2.12), (2.14), (2.15), (2.16) that for $\hat{\mathbb{M}}$ in some finite vicinity of zero the functions θ_j, a_j satisfy assumptions (A2.2), (A2.3) with $\delta_1 = 2\mu_1 - 2$, t_l, $l = 1, \dots, L$, being the collision times t_{ik}, $i, k = 1, \dots N$, $i \neq k$. So, lemma A2.1 implies lemma 2.1.

Proof of lemma A2.1. Lemma A2.1 follows from proposition 1.1 by a simple perturbation argument. On the intervals $[t_l, t_{l+1}]$, $l = 0, \dots L-1$ we introduce the following linear approximations $\theta^l(t)$, $a^l(t)$ of $\theta(t)$, $a(t)$:

$$\theta^l(t) = \theta(t) - \int_{t_l}^{t} ds \int_{t_l}^{s} ds_1 \left(1 - \eta(\frac{s_1 - t_l}{t_{l+1} - t_l})\right) \theta''(s_1)$$

$$- \int_{t}^{t_{l+1}} ds \int_{s}^{t_{l+1}} ds_1 \eta(\frac{s_1 - t_l}{t_{l+1} - t_l})\theta''(s_1),$$

$$a^l(t) = a(t) - \int_{t_l}^{t} ds \int_{t_l}^{s} ds_1 \left(1 - \eta(\frac{s_1 - t_l}{t_{l+1} - t_l})\right) a''(s_1)$$

$$- \int_{t}^{t_{l+1}} ds \int_{s}^{t_{l+1}} ds_1 \eta(\frac{s_1 - t_l}{t_{l+1} - t_l})a''(s_1).$$

Here $\eta \in C^\infty(\mathbb{R})$, $\eta(\xi) = \begin{cases} 1 \text{ for} |\xi| \le \frac{1}{4}, \\ 0 \text{ for} |\xi| \ge \frac{3}{4}. \end{cases}$

For $t \in [t_L, \infty)$ we define the corresponding $\theta^{L+1}(t)$, $a^{L+1}(t)$ as follows.

$$\theta^{L+1}(t) = \theta(t) - \int_t^\infty ds \int_s^\infty ds_1 \theta''(s_1),$$

$$a^{L+1}(t) = a(t) - \int_t^\infty ds \int_s^\infty ds_1 a''(s_1).$$

Clearly, for $t \in [t_l, t_{l+1}]$, $l = 0, \dots, L$, $t_{L+1} = \infty$, one has

$$|\theta(t) - \theta^l(t)|,\ |a(t) - a^l(t)| \le CR^{-2}, \tag{A2.4}$$

$$|\frac{d\theta^l}{dt}|,\ |\frac{da^l}{dt}| \le C(n + R^{-1}). \tag{A2.5}$$

On the interval $[t_l, t_{l+1}]$ one can pick out the leading term of (A2.1) in the form

$$i\chi_t = \mathcal{L}^l(t)\chi, \quad \mathcal{L}^l(t) = (-\triangle + E)\sigma_3 + \mathcal{V}^l(t), \tag{A2.6}$$

$$\mathcal{V}^l(t) = T^l(t)V(E^l)T^{l^*}, \quad T^l(t) = B_{\triangle^l(t), a^l(t), r^l},$$

$$\triangle^l(t) = \theta^l(t) - \frac{r^l \cdot a^l(t)}{2},\ r^l = \frac{da^l}{dt},\ E^l = E + \frac{d\theta^l}{dt} - \frac{|r^l|^2}{4}.$$

We denote the propagator corresponding to (A2.6) by $U^l(t,\tau)$. Clearly,

$$U^l(t,\tau) = T^l(t)e^{-i(t-\tau)L(E^l)}T^{l^*}(\tau), \quad P^l(t)U^l(t,\tau) = U^l(t,\tau)P^l(\tau),$$

where $P^l(t) = T^l(t)\hat{P}(E^l)T^{l^*}(t)$.

Consider the expression $\chi(t) \equiv U(t,\tau)P(\tau)f$, $t_l \le \tau < t_{l+1}$.

For $t_l \le t \le t_{l+1}$ we write $\chi(t)$ as the sum $\chi = h + k$, $h(t) = P^l(t)\chi(t)$. Since $\chi(t) = P(t)\chi(t)$, the $2d+2$ dimensional component k is controlled by h:

$$\|e^{\gamma|x-a(t)|}k(t)\|_2 \le C(|\theta(t) - \theta^l(t)| + |a(t) - a^l(t)| + |r^l| + |E - E^l|)\|e^{-\gamma|x-a(t)|}\chi(t)\|_2$$

$$\le C(R^{-1} + n)\|e^{-\gamma|x-a(t)|}\chi(t)\|_2 \tag{A2.7}$$

for some $\gamma > 0$, provided n, R^{-1} are sufficiently small. In the last inequality we used (A2.4), (A2.5).

For h one can write the following integral representation

$$h(t) = P^l(t)h_0(t) - i\int_\tau^t ds P^l(t)U^l(t,s)[\mathcal{V}^l(s)h_0(s) + R^l(s)\chi(s)], \tag{A2.8}$$

where

$$h_0(t) = e^{i(\triangle - E)(t-\tau)\sigma_3}P(\tau)f,$$

$$R^l(t) = \mathcal{V}(t) - \mathcal{V}^l(t) + i[P'(t), P(t)].$$

Obviously,

$$|\mathcal{V}(x,t)-\mathcal{V}^l(x,t)| \le C|\theta(t)-\theta^l(t)|+|a(t)-a^l(t)|+|r^l|+|E-E^l|)e^{-\gamma|x-a(t)|}$$

$$\le C(R^{-1}+n)e^{-\gamma|x-a(t)|}, \tag{A2.9}$$

$$|[P'(t),P(t)]f| \le Cne^{-\gamma|x-a(t)|}\|e^{-\gamma|x-a(t)|}f\|_2. \tag{A2.10}$$

Estimates (A2.7), (A2.9), (A2.10) and representation (A2.8) together with proposition 1.1 imply immediately that for $t_l \le \tau \le t \le t_{l+1}$ and for any $\xi \in \mathbb{R}$ the following inequality holds

$$\| \langle x-a(t)\rangle^{-\nu_0} \chi(t)\|_2 < t-\tau+\xi >^{d/2}$$

$$\le C \sup_{\tau\le s\le t} \left(\| \langle x-a(s)\rangle^{-\nu_0} e^{it\triangle\sigma_3(s-\tau)}P(\tau)f\|_2 < s-\tau+\xi >^{d/2}\right), \tag{A2.11}$$

where C do not depend on ξ. (A2.11) implies in particular, that

$$\| \langle x-a(t)\rangle^{-\nu_0} U(t,\tau)P(\tau)f\|_2 \le C < t-\tau >^{-d/2} \| \langle x-x_0\rangle^{\nu_0} f\|_2, \tag{A2.12}$$

$x_0 \in \mathbb{R}^d$, $t_l \le \tau \le t \le t_{l+1}$, $l=0,\dots,L$.

To prove that this estimate is in fact true for any $0 \le \tau \le t$ we use the induction arguments. Assume that one has (A2.12) for $\tau \le t \le t_l < \infty$. We need to show that then the same is true for $\tau \le t_l < t \le t_{l+1}$. For $t \in (t_l, t_{l+1}]$ we write $U(t,\tau)P(\tau)f = U(t,t_l)U(t_l,\tau)P(\tau)f$. Using (A2.12) and the representation

$$U(t,\tau)P(\tau)f = e^{i(t-\tau)(\triangle-E)\sigma_3}P(\tau)f$$

$$-i\int_\tau^t ds e^{i(t-s)(\triangle-E)\sigma_3}\left(\mathcal{V}(s)+i[P'(s),P(s)]\right)U(s,\tau)P(\tau)f, \tag{A2.13}$$

one checks easily that

$$\| < x-a(t) >^{-\nu_0} e^{i\triangle\sigma_3(t-t_l)}U(t_l,\tau)P(\tau)f\|_2 \le C < t-\tau >^{-d/2} \| \langle x-x_0\rangle^{\nu_0} f\|_2.$$

By (A2.11), this implies that (A2.12) is valid for $0 \le \tau \le t \le t_{l+1}$ and thus, for any $0 \le \tau \le t$. Moreover, by (A2.13) one can replace $a(t)$ in the left hand side of (A2.12) by any $x_1 \in \mathbb{R}^d$:

$$\| \langle x-x_1\rangle^{-\nu_0} U(t,\tau)P(\tau)f\|_2 \le C\| \langle x-x_0\rangle^{\nu_0} f\|_2, \quad 0 \le \tau \le t. \qquad \blacksquare$$

Appendix 3

Here we prove lemma 2.2. We start by proving a similar result for the "free" operators $T^0_{jkl}(t,\tau)$:

$$T^0_{jkl}(t,\tau) = A_j(t)\int_\tau^t d\rho\, \mathcal{U}_0(t,\rho)\tilde{\mathcal{V}}_k(\rho)\mathcal{U}_0(\rho,s)A_i(\tau).$$

Lemma A3.1. *For $i \neq k$, $t \geq 0$, one has*

$$\int_0^t d\tau \|T^0_{jki}(t,\tau)\| \leq W(\mathbb{M})\epsilon_{ik}^{\nu_1} \tag{A3.1}$$

with some $\nu_1 > 0$.

Proof. Since

$$\|T^0_{jkl}(t,\tau)\| \leq C < t-\tau >^{-d/2},$$

one has

$$I(t) = \int_0^t d\tau \|T^0_{jki}(t,\tau)\| \leq C\frac{t}{< t >}. \tag{A3.2}$$

In this appendix the constants C depend only on E_{0k}.

For $t \geq 2\rho$, where ρ is a small positive number, we write the integral $I(t)$ as a sum of two terms $I(t) = I_0(t) + I_1(t)$,

$$I_0(t) = \int_0^{t-2\rho} d\tau \|T^\rho_{jki}(t,\tau)\|,$$

$$T^\rho_{jki}(t,\tau) = A_j(t)\int_{\tau+\rho}^{t-\rho} ds\, \mathcal{U}_0(t,s)\tilde{\mathcal{V}}_k(s)\mathcal{U}_0(s,\tau)A_i(\tau),$$

$I_1(t)$ being the rest. Obviously,

$$I_1(t) \leq C\rho. \tag{A3.3}$$

Consider $I_0(t)$. To estimate this expression we write $T^\rho_{jki}(t,\tau)$ in the form

$$T^\rho_{jki} = \begin{pmatrix} \mathcal{T}^{11}_{jki} & \mathcal{T}^{12}_{jki} \\ -\mathcal{T}^{21}_{jki} & -\mathcal{T}^{22}_{jki} \end{pmatrix},$$

where

$$\mathcal{T}^{11}_{jki}(t,\tau) = A_j(t)\int_{\tau+\rho}^{t-\rho} ds\, e^{i(t-s)\triangle}\mathcal{V}^1_k(s)e^{i(s-\tau)\triangle}A_i(\tau),$$

$$\mathcal{T}^{12}_{jki}(t,\tau) = A_j(t)\int_{\tau+\rho}^{t-\rho} ds\, e^{i(t-s)\triangle}\mathcal{V}^2_k(s)e^{-i(s-\tau)\triangle}A_i(\tau),$$

$$\mathcal{V}^1_k(x,t) = V_1(x-\tilde{b}_k, E_0), \quad \mathcal{V}^2_k(x,t) = e^{2i\tilde{\Phi}_k(x,t)}V_2(x-\tilde{b}_k(t), E_0),$$

$$\mathcal{T}^{22}_{jki}(t,\tau)f = \overline{\mathcal{T}^{11}_{jki}(t,\tau)\bar{f}}, \quad \mathcal{T}^{21}_{jki}(t,\tau)f = \overline{\mathcal{T}^{12}_{jki}(t,\tau)\bar{f}} \tag{A3.4}$$

Consider $\mathcal{T}^{11}_{jki}(t,\tau)$. Since Hilbert-Schmidt norms dominate operator norms, we have

$$\|\mathcal{T}^{11}_{jki}(t,\tau)\|^2 \le C \int\limits_{\mathbb{R}^{2d}} dxdy < x >^{-2\nu} < y >^{-2\nu} |\mathcal{B}^1_{jki}(t,\tau)|^2,$$

where

$$\mathcal{B}^1_{jki}(t,\tau) = \int_{\tau+\rho}^{t-\rho} ds(t-s)^{-d/2}(s-\tau)^{-d/2}$$

$$\times \int\limits_{\mathbb{R}^d} dz e^{\frac{i|x-z+b_j(t)-\tilde{b}_k(s)|}{4(t-s)}} V_1(z) e^{\frac{i|z-y+\tilde{b}_k(s)-b_i(\tau)|}{4(s-\tau)}}.$$

Integrating by parts in the second integral and taking into account (2.13) one gets immediately the estimate

$$|\mathcal{B}^1_{jki}(t,\tau)| \le W(\mathbb{M})(< x > + < y >)\rho^{-1}$$

$$\times \int\limits_{\tau+\rho}^{t-\rho} ds(t-s)^{-d/2}(s-\tau)^{-d/2} < d_{jki}(t,s,\tau) >^{-1},$$

where

$$d_{jki}(t,s,\tau) = \frac{\tilde{b}_{jk}(t)}{(t-s)} + \frac{\tilde{b}_{ik}(\tau)}{(s-\tau)}.$$

Here the function W do not depend on ρ. As a consequence, one has for $0 \le \alpha < \min\{1, \frac{d}{4} - \frac{1}{2}\}$, $|\tilde{b}_{jk}(t)| + |\tilde{b}_{ik}(\tau)| > 0$,

$$\|\mathcal{T}^{11}_{jki}(t,\tau)\| \le W(\mathbb{M})\rho^{-1-d+2\alpha} \int\limits_{\tau}^{t} ds < t-s >^{-d/2+\alpha} < s-\tau >^{-d/2+\alpha}$$

$$\times |\tilde{b}_{jk}(t)(s-\tau) + \tilde{b}_{ik}(\tau)(t-s)|^{-\alpha} \le W(\mathbb{M})\rho^{-1-d+2\alpha}$$

$$\text{(A3.5)} \qquad \times < t-\tau >^{-d/2+2\alpha} (|\tilde{b}_{jk}(t) - \tilde{b}_{ik}(\tau)| + (t-\tau)|\tilde{b}_{jk}(t)|)^{-\alpha}.$$

Here we made use of the obvious inequality

$$\int\limits_{\mathbb{R}} ds < s >^{-a} < s-\rho >^{-a} |d_1 s + d_2|^{-\alpha} \le C < \rho >^{-a+\alpha} (|d_1| + |d_2|)^{-\alpha},$$

provided $a > 1$, $0 \le \alpha < 1$, d_1, $d_2 \in \mathbb{R}^d$, C being independent of d_1, d_2.

Integrating (A3.5) and taking into account (2.14,15) one gets finally,

$$\text{(A3.6)} \qquad \int_0^{t-2\rho} d\tau \|\mathcal{T}^{11}_{jki}(t,\tau)\| \le W(\mathbb{M})\rho^{-1-d+2\alpha}\epsilon^{\alpha}_{ik}.$$

In a similar way, one has for $\mathcal{T}^{12}_{jki}(t,\tau)$

$$\|\mathcal{T}^{12}_{jki}(t,\tau)\|^2 \le C \int\limits_{\mathbb{R}^{2d}} dxdy < x >^{-2\nu} < y >^{-2\nu} |\mathcal{B}^2_{jki}(t,\tau)|^2,$$

$$\mathcal{B}^2_{jki}(t,\tau) = \int_{\tau+\rho}^{t-\rho} ds (t-s)^{-d/2}(s-\tau)^{-d/2}$$

$$\times \int\limits_{\mathbb{R}^d} dz e^{\frac{i|x-z+b_j(t)-\tilde{b}_k(s)|}{4(t-s)}} e^{2i\tilde{\Phi}(z+\tilde{b}_k(s),s)} V_2(z) e^{-\frac{i|z-y+\tilde{b}_k(s)-b_i(\tau)|}{4(s-\tau)}},$$

which implies

$$\int_0^{t-2\rho} \|\mathcal{T}^{12}_{jki}(t,\tau)\| \leq W(\mathbb{M})\rho^{-1-d+2\alpha} \int\limits_0^t d\tau \int\limits_\tau^t ds < t-s >^{-d/2+\alpha}$$

$$\times < s-\tau >^{-d/2+\alpha} |\tilde{b}_{jk}(t)(s-\tau) - \tilde{b}_{ik}(\tau)(t-s)|^{-\alpha}$$

$$\leq W(\mathbb{M})\rho^{-1-d+2\alpha}\epsilon^{\alpha}_{ik}. \tag{A3.7}$$

Combining (A3.2), (A3.3), (A3.4), (A3.6), (A3.7) one obtains

$$I(t) \leq W(\mathbb{M})(\rho + \rho^{-1-d+2\alpha}\epsilon^{\alpha}_{ik}),$$

which leads immediately to (A3.1) with $\nu_1 \leq \frac{\alpha}{2+d-2\alpha}$. ■

Let us introduce the operators $T^1_{jki}(t,\tau)$:

$$T^1_{jkl}(t,\tau) = A_j(t) \int_\tau^t ds \mathcal{U}_0(t,s)\tilde{\mathcal{V}}_k(s)(I - P^A_k(s))\mathcal{U}_0(s,\tau)A_i(\tau).$$

It is not difficult to check that for any $\alpha \leq 1$,

$$\|A_j(t)\mathcal{U}_0(t,s)\tilde{\mathcal{V}}_k(s)(I - P^A_k(s))\mathcal{U}_0(s,\tau)A_i(\tau)\|$$

$$\leq W(\mathbb{M}) < t-s >^{-d/2} < s-\tau >^{-d/2+\alpha} < \tilde{b}_{ik}(\tau) >^{-\alpha}.$$

As a consequence,

$$\int_0^t d\tau \|T^1_{jkl}(t,\tau)\| \leq W(\mathbb{M}) \int_0^t d\tau < t-\tau >^{-d/2+\alpha} < \tilde{b}_{ik}(\tau) >^{-\alpha} \leq W(\mathbb{M})\epsilon^{\alpha}_{ik}. \tag{A3.8}$$

At the last step here we have used (2.14), (2.15).

Proof of lemma 2.2. This lemma follows directly from (A3.1), (A3.8) and the following representation

$$T_{jkl}(t,\tau) = T^0_{jki}(t,\tau) - T^1_{jki}(t,\tau)$$

$$-i \int_\tau^t d\rho \int_\tau^\rho ds A_j(t)\mathcal{U}_0(t,\rho)\tilde{\mathcal{V}}_k(\rho)P^A_k(\rho)\mathcal{U}^A_k(\rho,s)R_k(s)\mathcal{U}_0(s,\tau)A_i(\tau) \tag{A3.9}$$

$$-i \int_\tau^t d\rho A_j(t)\mathcal{U}_0(t,\rho)\tilde{\mathcal{V}}_k(\rho)P^A_k(\rho)A^{-1}_k(\rho)T^0_{kki}(\rho,\tau) \tag{A3.10}$$

(A3.11)
$$-\int_\tau^t d\rho\int_\tau^\rho ds A_j(t)\mathcal{U}_0(t,\rho)\tilde{\mathcal{V}}_k(\rho)P_k^A(\rho)\mathcal{U}_k^A(\rho,s)[\tilde{\mathcal{V}}_k(s)+R_k(s)]A_k^{-1}(s)T_{kki}^0(s,\tau).$$

We estimate the right hand side of this representation term by term. Using lemma 2.1 and inequality (2.21) one gets

$$\int_0^t d\tau\|(A3.9)\|\le W(\hat{\mathbb{M}})\mathbb{M}_0(t). \tag{A3.12}$$

Expression (A3.10) can be estimated as follows
(A3.13)
$$\int_0^t d\tau\|(A3.10)\|\le W(\mathbb{M})\int_0^t d\tau\int_\tau^t d\rho <t-\rho>^{-d/2}\|T_{kki}^0(\rho,\tau)\|\le W(\mathbb{M})\epsilon_{ik}^\alpha.$$

In a similar way,

$$\int_0^t d\tau\|(A3.11)\|\le W(\hat{\mathbb{M}})\int_0^t d\tau\int_\tau^t d\rho\int_\tau^\rho ds <t-\rho>^{-d/2}$$

$$<\rho-s>^{-d/2}\|T_{kki}^0(s,\tau)\|\le W(\hat{\mathbb{M}})\epsilon_{ik}^\alpha. \tag{A3.14}$$

Combining (A3.1), (A3.8), (A3.12), (A3.13), (A3.14) one gets lemma 2.2. ∎

Appendix 4

Here we discuss the proof of proposition 1.1. Since only the weighted estimates are needed, rather then follow [8, 37, 38] we use the approach of [16, 17, 18]. It turns out that the arguments of [16, 17, 18] can be applied almost without modifications. So, we describe only the main steps of the proof, referring the reader to [16, 17, 18] for most of the details.

We start be recalling briefly some basic properties of the free resolvent $R_0(\lambda)=\begin{pmatrix}(-\triangle+E-\lambda)^{-1} & 0\\ 0 & -(-\triangle+E+\lambda)^{-1}\end{pmatrix}$. Let $H^{t,s}$ stand for the weighted Sobolev spaces:

$$H^{t,s}=\{f,\ \|f\|_{H^{t,s}}\equiv\|<x>^s(1-\triangle)^{t/2}f\|_2<\infty\}.$$

We denote by $B(H^{s,t},H^{s_1,t_1})$ the space of bounded operators from $H^{s,t}$ to H^{s_1,t_1}. Set $L_2^s=H^{0,s}$, $B(H^{s,t})=B(H^{s,t},H^{s,t})$. If $s>1$ and $t\in\mathbb{R}$ the resolvent $R_0(\lambda)$ which is originally defined as $B(L_2)$ valued analytic function of $\lambda\in\mathbb{C}\setminus(-\infty,-E]\cup[E,\infty)$ can be extended continuously to the $\overline{\mathbb{C}^+}=\{\operatorname{im}\lambda\ge 0\}$ when considered as a $B(H^{s,t},H^{-s,t+2})$ valued function. The following properties of $R_0(\lambda)$ are well known, see [16, 17, 18, 37, 38] and references therein.

Lemma A4.1. *Let $k=0,1,\dots$. If $s>k+1/2$, then the derivative $R_0^{(k)}(\lambda)\in B(H^{s,0},H^{-s,0})$ is continuous in $\lambda\in\overline{\mathbb{C}^+}\setminus\{E,-E\}$, with*

$$R_0^{(k)}(\lambda)=O(|\lambda|^{-(k+1)/2}), \tag{A4.1}$$

in this norm as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$.

The behavior of $R_0(\lambda)$ for λ close to $\pm E$ is described by the following lemma, see again [16,17,18].

Lemma A4.2. *As $\lambda \to E$, $R_0(\lambda)$ admits the following asymptotic expansion in $B(H^{s,t}, H^{-s,t+2})$.*
For m odd:

$$(A4.2) \qquad R_0(\lambda) = \sum_{j=0}^{l} G_{j,0}(\lambda - E)^j + \sum_{j=0}^{l} G_{j,1}(\lambda - E)^{j+\frac{1}{2}} + O((\lambda - E)^{l+1}),$$

for m even:

$$(A4.3) \qquad R_0(\lambda) = \sum_{j=0}^{l} G_{j,0}(\lambda - E)^j + \ln(\lambda - E) \sum_{j=0}^{l} G_{j,1}(\lambda - E)^j + o((\lambda - E)^l),$$

where $l = 0, 1, \dots,$ $s > C(l,d)$, the coefficients $G_{j,k}$ belong to $B(H^{s,t}, H^{-s,t+2})$, $G_{j,1} = 0$ for $j < \frac{d-3}{2}$ if d is odd and for $j < \frac{d-2}{2}$ if d is even. Representations (A4.2), (A4.3) *can be differentiated with respect to λ any number of times.*

Here $(\lambda - E)^{1/2}$ and $\ln(\lambda - E)$ are defined on the complex plane with the cut along $[E, \infty)$. The explicit expressions for the constants $C(l,d)$ can be found in [17, 18, 19]. Similar expansions hold as $\lambda \to -E$.

For $\lambda \in [E, \infty)$, consider the operator

$$I + R_0(\lambda + i0)V : L_2^{-s} \to L_2^{-s},$$

$s > 1$.

Lemma A4.3. *Let $E \in \mathcal{A}_0$. Then* $\mathrm{Ker}(I + R_0(\lambda + i0)V)$ *is trivial.*

Proof. We start by the case $\lambda = E$. Let $\psi \in \mathrm{Ker}(I + G_0 V)$. This implies that ψ belongs to $L_2(\mathbb{R}^d) + < x >^{-(d-2)} L_\infty(\mathbb{R}^d)$ and satisfies

$$L\psi = E\psi.$$

Hypothesis H3 then allows us to conclude that $\psi = 0$.

We consider next the case $\lambda > E$. Let $\psi \in \mathrm{Ker}(I + R_0(\lambda + i0)V)$. Since V is spherically symmetric, one can assume that $\psi(x) = f(r)Y(\omega)$, $r = |x|$, $\omega = \frac{x}{|x|}$, $f \in L_2(\mathbb{R}_+; r^{d-1} < r >^{-2s} dr)$ and $Y \in L_2(S^{d-1})$,

$$\triangle_{S^{d-1}} Y = \mu_n Y, \quad \mu_n = n(d - 2 + n),$$

for some $n \in \{0, 1, \dots\}$. Then f has to satisfy

$$(A4.4) \qquad l_n f \equiv \left[(-\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + E + \frac{\mu_n}{r^2})\sigma_3 + V \right] f = \lambda f,$$

$$f'(0) = 0 \text{ if } \quad n = 0, \quad f(0) = 0 \text{ if } \quad n > 0,$$

and as $r \to \infty$,

$$(A4.5) \qquad f = cr^{-\frac{(d-2)}{2}} H_\nu^{(1)}(kr) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O(e^{-\gamma r}), \quad \gamma > 0,$$

for some constant c. Here $k=(\lambda-E)^{1/2}>0$, $\nu=n+\frac{(d-2)}{2}$, $H_\nu^{(1)}$ is the first Hankel function. Asymptotic representation (A4.5) can be differentiated with respect to r any number of times.

The Wronskian

$$w(f,g)=r^{d-1}(<f',g>_{\mathbb{R}^2}-<f,g'>_{\mathbb{R}^2})$$

does not depend on r if f and g are solutions of (A4.4). Calculating $w(f,\bar{f})$ one gets

$$2ik|c|^2=0,$$

which implies that $\psi\in L_2$. Since $E\in\mathcal{A}_0$, this means that $\psi=0$. ■

Consider the full resolvent $R(\lambda)=(L-\lambda)^{-1}$. $R(\lambda)\hat{P}$ $(R(\lambda))$ is a $B(L_2)$ valued holomorphic (meromorphic with the only pole in zero) function of $\lambda\in\mathbb{C}\setminus(-\infty,-E]\cup[E,\infty)$. $R(\lambda)$ satisfies the relations

$$\sigma_1R(\lambda)\sigma_1=-R(-\lambda). \tag{A4.6}$$

The analytic properties of $R(\lambda)$ near the cuts $(-\infty,-E]$, $[E,\infty)$ are collected in the two following lemmas. In both of them we assume that $E\in\mathcal{A}_0$.

Lemma A4.4. *For $s>1$, $R(\lambda)\hat{P}$ can be extended continuously to $\overline{\mathbb{C}^+}$ as a $B(L_2^s,L_2^{-s})$ valued function. Moreover, if $s>k+\frac{1}{2}$ then $R^{(k)}(\lambda)\hat{P}$ exists and continuous for $\lambda\in\overline{\mathbb{C}^+}\setminus\{E,-E\}$ and*

$$R^{(k)}(\lambda)\hat{P}=O(|\lambda|^{-(k+1)/2}) \tag{A4.7}$$

in $B(L_2^s,L_2^{-s})$ as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$.

Lemma A4.5. *As $\lambda\to E$, $R(\lambda)$ admits the following asymptotic expansion in $B(L_2^s,L_2^{-s})$.*
For m odd:

$$R(\lambda)=\sum_{j=0}^{l}B_{j,0}(\lambda-E)^j+\sum_{j=0}^{l-1}B_{j,1}(\lambda-E)^{j+\frac{1}{2}}+O((\lambda-E)^l), \tag{A4.8}$$

for m even:

$$R(\lambda)=\sum_{j=0}^{l}\sum_{k=0}^{\infty}B_{j,k}(\lambda-E)^j(\ln(\lambda-E))^k++o((\lambda-E)^l), \tag{A4.9}$$

where $l=0,1,\dots,$ $s>C(l,d)$, $B_{j,k}\in B(L_2^s,L_2^{-s})$, $B_{j,k}=0$ for $k=1$, $j<\frac{d-3}{2}$ if d is odd and for $k>\frac{2j}{d-2}$ if d is even. Representations (A4.8), (A4.9) *can be differentiated with respect to λ any number of times.*

These results is a standard consequence of the corresponding properties of the free resolvent (lemmas A4.1,2) and lemma A4.3, see [16, 17, 18].

Consider the propagator e^{-itL}. Lemma A4.4, together with (A4.6), (A4.8), (A4.9) allows us to represent the expression $\left\langle e^{-itL}\hat{P}f,g\right\rangle$, $f,g\in C_0^\infty(\mathbb{R}^d)$ in the form

$$\left\langle e^{-itL}\hat{P}fg\right\rangle=\int_E^\infty d\lambda[e^{-i\lambda t}\left\langle\mathcal{E}(\lambda)f,g\right\rangle-e^{i\lambda t}\left\langle\mathcal{E}(\lambda)\sigma_1f,\sigma_1g\right\rangle], \tag{A4.10}$$

where

$$\mathcal{E}(\lambda) = \frac{1}{2\pi i}(R(\lambda + i0) - R(\lambda - i0)).$$

It follows from (A4.8), (A4.9) that as $\lambda \to E$, $\mathcal{E}(\lambda)$ admits the following asymptotic expansion in $B(L_2^s, L_2^{-s})$ with s sufficiently large.
For d odd:

$$\mathcal{E}(\lambda) = \mathcal{E}_0(\lambda - E)^{\frac{d-2}{2}} + O((\lambda - E)^{\frac{d}{2}}), \tag{A4.11}$$

for d even:

$$\mathcal{E}(\lambda) = \mathcal{E}_0(\lambda - E)^{\frac{d-2}{2}} + \begin{cases} O(\ln(\lambda - E)(\lambda - E)^2) & \text{if } d = 4, \\ O((\lambda - E)^{\frac{d}{2}}) & \text{if } d \geq 6. \end{cases} \tag{A4.12}$$

$\mathcal{E}_0 \in B(L_2^s, L_2^{-s})$. These expansions can be differentiated with respect to λ any number of times.

Combining (A4.10), (A4.7), (A4.11), (A4.12) one gets immediately [18]

$$\| < x >^{-s} e^{-itL} \hat{P} f\|_2 \leq C < t >^{-d/2} \| < x >^s f\|_2,$$

provided s is sufficiently large. To recover proposition 1.1 it is sufficient now to inject this inequality in the following representation for $e^{-itL}\hat{P}$

$$e^{-itL}\hat{P} = \hat{P}e^{-itL_0} - i\int_0^t ds e^{-i(t-s)L_0}\hat{P}Ve^{-isL_0}$$

$$-\int_0^t ds \int_s^t d\rho e^{-i(t-\rho)L_0} V e^{-i(\rho-s)L}\hat{P}Ve^{-isL_0}.$$

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References.

1. Benjamin, T.B. The stability of solitary waves. Proc. Roy. Soc. Lond. **1972**, A*328*, 153-183.
2. Berestycki, H.; Lions, P.-L. Nonlinear scalar field equations, I, II, Arch. Rat. Mech. Anal. **1983**, *82* (4), 313-375.
3. Bourgain, J.; Wang, W. Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **1997**, *25* (4), 197-215.
4. Buslaev V.S.; Perelman, G.S. Scattering for the nonlinear Schrödinger equation: states close to a soliton. St. Petersburg Math. J. **1993**, *4* (6),1111-1143.
5. Buslaev,V.S.; Perelman, G.S. On the stability of solitary waves for nonlinear Schrodinger equation. Amer. Math. Soc. Transl.(2). **1995**, *164*, 75-99.
6. Buslaev, V.S.; Sulem, C. On the asymptotic stability stability of solitary waves of nonlinear Schrodinger equations. Preprint.

7. Cazenave, T.; P.-L.Lions, P.-L. Orbital stability of standing waves for some nonlinear Schrödinger equations, Commun. Math. Phys. **1982**, *85* (4), 549-561.

8. Cuccagna, S. Stabilization of solutions to nonlinear Schrödinger equation, Comm. Pure Appl. Math. **2001** *54*, 1110-1145.

9. Cuccagna, S. On asymptotic stability of ground states of NLS. Preprint.

10. Ginibre, J.; Velo G. On a class of nonlinear Schrödinger equations I, II. J.Func.Anal. **1979**, *32*, 1-71.

11. Ginibre, J.; Velo G. On a class of nonlinear Schrödinger equations III. Ann. Inst. H.Poincare -Phys. Theor. **1978**, *28* (3), 287-316.

12. Grikurov, V. Preprint, 1995.

13. Grillakis, M.; Shatah, J.; Strauss, W. Stability of solitary waves in presence of symmetry I. J. Func. Anal. **1987**, *74* (1), 160-197.

14. Grillakis, M.; Shatah J.; Strauss, W. Stability of solitary waves in presence of symmetry II. J. Func. Anal. **1990**, *94* (2), 308-384.

15. Hagedorn, G. Asymptotic completeness for the impact parameter approximation to three particle scattering. Ann. Inst. Henri Poincaré. **1982**, *36* (1), 19-40.

16. Jensen, A. Spectral properties of Schrödinger operators and time decay of the wave functions. Results in $L_2(\mathbb{R}^m)$, $m \geq 5$. Duke Math. J. **1982**, *47* (1), 57-80.

17. A.Jensen, A. Spectral properties of Schrödinger operators and time decay of the wave functions. Results in $L_2(\mathbb{R}^4)$. J. Math. Anal. Appl. **1984**, *101* (2), 397-422.

18. A.Jensen, A.; Kato, T. Spectral properties of Schrödinger operators and time decay of the wave functions. Duke Math. J. **1979**, *46* (3), 583-611.

19. Martel, Y.; F.Merle, F.; Tsai, T.-P. Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. Comm. Math. Phys. **2001**, *231*, 347-373.

20. McLeod, K. Uniqueness of positive radial solutions of $\triangle u + f(u) = 0$ in $\mathbb{R}^n$. Trans. Amer. Math. Soc. **1993**, *339* (2), 495-505.

21. Nier, F.; Soffer, A. Dispersion and Strichartz estimates for some finite rank perturbations of the Laplace operator. J. of Func. Analysis, to appear.

22. Novikov S.P.(ed.), Theory of solitons: The inverse scattering method, Moscow, Nauka, 1980.

23. Perelman, G. Some results on the scattering of weakly interacting solitons for nonlinear Schrödinger equation. In: Spectral Theory, Microlocal Analysis, Singular Manifolds, M.Demuth et al., eds., Math. Top. 14, Berlin, Akademie Verlag, 1997, pp. 78-137.

24. Pillet, C.-A.; Wayne, C.E.Invariant manifolds for a class of dispersive, Hamiltonian, partial differential equations. J. Diff. Eq. **1997**, *141* (2), 310-326.

25. Reed, M.; Simon, B. Methods of modern mathematical physics II: Scattering theory, New York, Academic Press, 1979.

26. Shatah, J; Strauss, W. Instability of nonlinear bounded states. Commun. Math. Phys. **1987**, *100* (2), 35-108.

27. Soffer A.; Weinstein, M.I. Multichannel nonlinear scattering theory for nonintegrable equations I. Commun. Math. Phys. **1990**, *133* (1), 119-146.

28. Soffer A.; Weinstein, M.I. Multichannel nonlinear scattering theory for nonintegrable equations II. J. Diff. Eq. **1992**, *98* (2), 376-390.

29. Sulem, C.; Sulem, P.-L. The nonlinear Schrödinger eqation. Self-focusing and wave collapse, Appl.Math. Sciences, 139. Springer, New Yowrk, 1999.

30. Tsai, T.-P.; H.-T. Yau, H.-T. Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and dispersion dominated solutions. Comm. Pure Appl. Math. **2002**, 55, 153-216.

31. Tsai, T.-P.; Yau, H.-T. Relaxation of exited states in nonlinear Schrödinger equations. IMRN, to appear.

32. Tsai, T.-P.; Yau, H.-T. Stable directions for exited states of nonlinear Schrödinger equations. Comm. PDE, to appear.

33. Tsai, T.-P.; Yau, H.-T. Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data. Preprint.

34. Weinstein, M.I. Modulation stability of ground states of nonlinear Schrödinger equations. SIAM J. Math. Anal. **1985**, *16* (3), 472-491.

35. Weinstein, M.I. Lyapunov stability of ground states of nonlinear dispersive evolution equations. Comm. Pure Appl. Math. **1986**, *39* (1), 51-68.

37. Weder, R. Center manifold for nonintegrable nonlinear Schrödinger equations on the line. Comm. Math. Phys. **2000**, *215* (2), 343-356.

38. Yajima, K. The $W^{k,p}$ continuity of wave operators for Schrödinger operators. J. Math. Soc. Japan **1995**, *47* (3), 551-581.

39. Yajima, K. The $W^{k,p}$ continuity of wave operators for Schrödinger operators. III. Even- dimensional cases $m \geq 4$. J. Math. Scien. Univ. Tokyo **1995**, *2* (20, 311-346.